Financial Implications of Models with Indeterminacy and Variable Capacity Utilization*

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Abstract

We study asset pricing in a one sector dynamic general equilibrium model with indeterminacy and variable utilization of capital. The indeterminacy results due to the presence of increasing returns to scale. The variable capacity utilization lowers the threshold value of increasing returns to scale at which multiple equilibria arise. It also promotes flexibility in the model, resulting in extremely smooth consumption. The model does not solve the equity premium or the volatility puzzle. The same is true when we add habit formation and/or costs of adjustment. We therefore conclude that this model cannot adequately represent the stylized financial facts.

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1 Introduction

Starting with the pioneering work of Benhabib and Farmer [3] and Farmer and Guo [10], a large body of literature has developed in which Real Business Cycle (RBC) models, modified to include increasing returns to scale in production, can result in a continuum of equilibria indexed by agents' expectations.[1] Models with multiple equilibria successfully replicate essential macroeconomic features of the business cycle. In this regard, these models are similar to the neoclassical RBC model. Prior research has shown that the standard RBC model fails when confronted with the financial market data.[2] In this paper, we examine the asset pricing implications of a model with indeterminacy. To our knowledge, this study is the first to undertake such an analysis.

While fluctuations in the standard RBC model are driven by technology shocks, in our model it is agents' expectations that cause business cycles. Since financial markets are theorized to be driven, at least in part, by agents' expectations, one might expect that this model would reflect well the behavior of such markets. If such an improvement is found, it would enhance support for the use of models with self-fulfilling expectations over traditional RBC models without indeterminacy.

In early versions of one-sector models with indeterminacy of equilibria, including Farmer and Guo [10], the size of increasing returns to scale required for indeterminacy is much higher than empirical estimates suggest. The following example illustrates: Suppose that agents expect the return on capital to increase next period. They will react by increasing next period's capital stock. Only with very high increasing returns to scale (around 1.5) will the marginal product of capital be increasing in capital and therefore will the expectation be fulfilled.

An alternative specification of the one-sector model, which exhibits indeterminacy with moderate increasing returns to scale (about 1.1), is found in Wen [27]. The model incorporates variable utilization of capital and is

[1] See Benhabib and Farmer [5] for an excellent survey of this literature.

[2] Kocherlakota [20] and Siegel and Thaler [24] survey the recent literature on asset pricing puzzles.

very successful in summarizing the key stylized macroeconomic facts. In two-sector versions (see for example Benhabib and Farmer [4] and Harrison [14]), the required returns to scale are also within an empirically acceptable range (about 1.1, which is well within the range of plausibility for sector-specific returns in Harrison [15]), as the shifting of resources between sectors affects the relative price of capital, boosting the return on capital. However, also as a result of the increased factor mobility, countercyclical consumption results as agents shift resources into the highly productive investment sector.

While our focus in this research is on financial returns in the context of self-fulfilling expectations, we wish to select an empirically plausible model with adequate macroeconomic performance as a vehicle for our study. We therefore use Wen's model as the benchmark in our exploration of the financial implications of models with multiple equilibria.

Taking the benchmark model to the financial data, we find an equity premium of 0.09 of one percent. This is 150 times higher than that produced by the standard RBC model; but it is negligible when compared with the equity premium in U.S. data. The risk-free rate is 4.1 percent, which is similar to the risk free-rate generated by traditional RBC models and is much higher than its empirical counterpart. In addition, returns on financial assets, especially the return on equity, do not exhibit realistic volatility. Our results show that although the model with self-fulfilling expectations marginally improves on the financial market performance of models without indeterminacy, this improvement is insufficient to judge it a success on this front.

The question arises whether the poor asset pricing results of the benchmark model can be improved with modifications which alleviated or resolved financial puzzles in the RBC model. For example, Jermann [17] incorporates habit formation and adjustment costs into a production economy with inelastic labor supply. With these additions, his model is able to match first moments of asset returns and the volatility of the return on equity, but the volatility of the risk-free rate is exaggerated. Avalos [1] combines habit formation with other kinds of adjustment costs, such as time to build or time to plan, and generates a time series of artificial asset returns which repli-

cates the expected returns in the U.S. data but significantly overstates their volatility. Boldrin, Christiano and Fisher [6] introduce habit persistence into a two-sector model with limited capital and labor mobility and obtain similar results.

The combination of habit persistence and capital adjustment costs resolves the asset pricing puzzles in production economies because the agent whose preferences display habit persistence is eager to avoid fluctuations in his consumption level. When frictions such as adjustment costs are present, the equity security becomes an unattractive instrument for consumption smoothing relative to the risk-free asset. As a result, the agent requires a higher return for holding equity and accepts a lower return on bonds. On the other hand, adjustment costs prevent the instantaneous response of the capital stock to exogenous shocks and therefore increase the volatility of the return on equity.

We therefore modify the benchmark model, introducing adjustment costs, habit formation and the combination thereof. First, we need to ensure that these modifications are consistent with indeterminacy in the models. Kim [18] and Wen [28] find that incorporating adjustment costs into the one-sector model without capacity utilization rules out indeterminacy at any realistic value of increasing returns. Our results show that the addition of habit formation, adjustment costs, or both to the benchmark model with variable utilization of the capital stock has negligible effect on the size of returns to scale necessary for indeterminacy. We therefore calibrate and simulate each of the models and compare their macroeconomic and financial properties.

As expected, the proposed modifications do not compromise the macroeconomic performance of the benchmark model. The financial implications, however, are more surprising. Our results show that unlike in traditional RBC models, the inclusion of habit persistence and adjustment costs does not help in matching the stylized asset pricing facts. We attribute this failure to the following tension: For swings in optimism and pessimism to translate into corresponding movements in economic activity there must be enough flexibility in the model economy to allow agents to act on their expecta-

tions. In our benchmark model the variable capacity utilization plays this role. This flexibility negates the effect of adjustment costs, allowing agents to smooth their consumption to the desired level.[3][4]

We therefore conclude that general equilibrium production models with self-fulfilling expectations cannot achieve an accurate representation of the stylized financial markets facts with empirically plausible increasing returns because of two conflicting requirements.[5] On the one hand, in order to generate a sufficient equity premium and volatility of the asset returns we need frictions to restrict the mobility of factors of production, especially of capital. On the other hand, we require flexibility in factor responses to shocks to lower the degree of the increasing returns to an empirically justifiable level.[6] Perhaps a model in which indeterminacy is introduced through channels other than aggregate increasing returns in production will fare better in this regard.[7]

The rest of this paper proceeds as follows: In Section 2 we describe the model and its equilibrium. In Section 3 we choose parameter values and present our results. In Section 4 we conclude.

2 The Model

Throughout the paper we work with the class of models based on a decentralized version of an RBC model with variable capacity utilization proposed by

[3] In fact the insufficient variability of consumption is the main macroeconomic drawback of Wen's model.

[4] To separate the effects of increasing returns to scale from those of variable utilization, we investigate analogous economies with constant returns to scale in production. In these economies, fluctuations are driven by persistent technology shocks. Consumption is extremely smooth in these models as well, and it therefore comes as no surprise that characteristics of financial returns are largely unchanged.

[5] Variable capacity utilization is only one way to generate indeterminacy with mild increasing returns to scale. Multiple sector production models achieve the same objective. However, these models are characterized by countercyclical consumption and their financial results are poor as well.

[6] We also considered standard one sector production model with multiple equilibria such as in Farmer and Guo [10]. We found that their financial statistics are similar to those from our Benchmark model. In addition, with capital adjustment costs, indeterminacy cannot result in Farmer and Guo's model with a realistic value of the externality.

[7] We do not consider monetary models and models with mark-ups in this research.

Greenwood *et al.* [12] and analyzed in the context of local indeterminacy by Wen [27]. We consider several model specifications with and without habit formation and costs of adjusting the capital stock. The economy is populated by a large number of identical infinitely-lived households and firms. There are two types of financial assets available: the risk-free one-period bond, paying one unit of the consumption good in every state of nature and the equity security, which is a claim to the output produced by firms after payments to factors of production. The firms produce with a constant returns to scale technology that may be subject to an externality, so that social returns may be increasing.

2.1 Households

There is a continuum of identical households indexed by [0,1]. Households maximize their expected lifetime utility over consumption and leisure by deciding on the time they wish to work and by choosing their financial asset holdings. The behavior of the representative household is characterized by the following maximization problem:

$$Max_{\{Z^e_{t+1}, Z^b_{t+1}, N^h_t\}} E\left(\sum_{t=0}^{\infty} \beta^t U(C^h_t, H_t, 1 - N^h_t)\right)$$

subject to:

$$C_t + P^e_t Z^e_{t+1} + P^b_t Z^b_{t+1} \leq (P^e_t + D_t) Z^e_t + Z^b_t + W_t N^h_t, \tag{1}$$

where β denotes the subjective discount factor, C^h_t is per capita consumption, N^h_t denotes labor services, and W_t is the competitively determined wage rate, each in period t. Each household is endowed with one unit of time. Z^e_t is the fraction of the single equity share and Z^b_t is the number of risk-free bonds held by the representative household in period t.[8] The period t prices of the equity security and risk-free asset are denoted by P^e_t and P^b_t respectively. The period t dividend, D_t, will be defined shortly.

[8] We do not consider leverage effects because they have a negligible effect on asset prices in our economy.

The period preference ordering of the representative household is assumed to be separable in consumption and leisure and has its origins in Hansen [13]:

$$U\left(C_t^h, H_t, N_t^h\right) = \frac{\left(C_t^h - H_t\right)^{1-\xi} - 1}{1-\xi} - \Lambda N_t^h \tag{2}$$

$\{H_t\}_{t=0}^{\infty}$ denotes a stochastic sequence of habits. If there is no habit formation, $\{H_t\}_{t=0}^{\infty} = 0$, the representative household's marginal utility of consumption is given by $MU_t^h = \left(C_t^h\right)^{-\xi}$ and its marginal rate of substitution in consumption simplifies to:

$$MRS_{t,t+j}^h = \beta^j \frac{U_1\left(C_{t+j}^h, N_{t+j}^h\right)}{U_1\left(C_t^h, N_t^h\right)} = \beta^j \left(\frac{C_{t+j}^h}{C_t^h}\right)^{-\xi} \tag{3}$$

We consider the habit formation specification proposed by Campbell and Cochrane [9], in which the individual household's habit is exogenously determined by the history of aggregate consumption and responds slowly to current consumption.

Let $SR_t = \frac{C_t - H_t}{C_t}$ be the surplus consumption ratio. For the remainder of this paper we denote by small letters log-deviations of variables from their steady stated values. For example, $sr_t = \ln SR_t - \ln \overline{SR}$, where $\overline{SR}$ is a fixed steady state parameter. Following Lettau and Uhlig [21] we adopt a linear version of the process describing the evolution of the consumption surplus ratio:

$$sr_{t+1} = \lambda sr_t + \left(\frac{1}{\overline{SR}} - 1\right)(c_{t+1} - c_t) \tag{4}$$

The linearity of equation (4) preserves the log-linear environment, which is essential for the solution method used in this paper.[9] With external habit, the representative household's marginal utility of consumption is $MU_t =$

[9] Lettau and Uhlig [21] show that linearization of the process governing the evolution of the consumption surplus ratio does not have dramatic implications for the risk-free rate, which does not change much and does not become very volatile. Linearization, however, does limit the rich asset pricing implications of Campbell and Cochrane [9].

$\left(SR_t C_t^h\right)^{-\xi}$ and its marginal rate of substitution in consumption is given by:

$$MRS_{t,t+j} = \beta^j \left(\frac{SR_{t+j}}{SR_t}\frac{C_{t+j}}{C_t}\right)^{-\xi} = \beta^j \exp\left(-\xi\left(\lambda - 1\right) sr_t + \frac{1}{\overline{SR}}\left(c_{t+1} - c_t\right)\right) \tag{5}$$

The first order conditions, derived from the household's optimization problem (1), produce two asset pricing equations:

$$P_t^e = E_t\left[MRS_{t,t+1}^h\left(P_{t+1}^e + D_{t+1}\right)\right] + e^{\nu_t} \tag{6}$$

$$P_t^b = E_t\left[MRS_{t,t+1}^h\right] \tag{7}$$

where ν_t is a non-fundamental belief or sunspot shock. This shock is an extra return on the equity security (in terms of a utility increment), which the household believes to materialize over the period. Under certain parameterizations of the model, beliefs become self-fulfilling. This shock is known at time t. Therefore, the expectations in equations (6) and (7) are point expectations. Dynamically, however, there is an uncertainty related to the sunspot shock because future realizations of the sunspot shock are not known. Consistency with rational expectations requires:

$$E\left[\nu_t\right] = 0.$$

The first order condition for the labor decision is:

$$W_t = \frac{\Lambda}{MU_t^h} \tag{8}$$

2.2 Firms

The production sector consists of a continuum of identical firms. The production function of each individual firm is:

$$Y_t = A_t X_t \left(U_t^f K_t^f\right)^{\alpha}\left(N_t^f\right)^{1-\alpha} \tag{9}$$

The variable A_t in equation (9) is the level of technology, the log of which is assumed to follow an AR(1) process:

$$\ln A_t = (1-\rho)\ln \overline{A} + \rho \ln A_{t-1} + \varepsilon_t. \tag{10}$$

with $\rho \in (0,1)$. $U_t^f \in (0,1)$ represents the rate of capacity utilization of the capital stock K_t^f. N_t^f is the labor input. There is an external effect, X_t, which is taken as given by the individual firms, so that each behaves competitively. It is this externality that allows the social production function to exhibit increasing returns to scale while private returns are constant. The production externality is given by:

$$X_t = \left(\overline{U}_t \overline{K}_t\right)^{\alpha\eta} \overline{N}_t^{(1-\alpha)\eta}$$

Variables with bars denote economy-wide quantities. The parameter η captures the size of the production externality. If $\eta = 0$, there is no externality and returns to scale are constant at both the private and social level.

Each firm begins period t with the stock of capital K_t^f carried over from the previous period. The evolution of the capital stock is given by:

$$K_t^f = \left(1 - \Omega_{t-1}^f\right) K_{t-1}^f + I_{t-1}^f, \tag{11}$$

where I_t^f is period t investment and Ω_t^f is the period's depreciation rate. The rate of capital depreciation is assumed to be an increasing function of the capacity utilization rate. Thus the depreciation rate is defined endogenously as:

$$\Omega_t^f = \tau \left(U_t^f\right)^\theta$$

where $0 < \tau < 1$ and $\theta > 1$.

A representative firm produces and sells output and uses proceeds of the output sale to pay the wage bill, $W_t N_t^f$ and to finance investments, I_t^f, under the knowledge of the equation of motion of the capital stock (11). The remaining output is distributed as dividends:

$$D_t^f = Y_t^f - W_t N_t^f - I_t^f - g(I_t - \tau U_t^\theta K_t), \tag{12}$$

where $g(\bullet)$ is a concave function, which captures the cost of adjustment of capital stock. We use the symmetric quadratic function of net investment $g_t = \frac{\phi}{2K_t}\left(I_t - \tau U_t^\theta K_t\right)^2$.[10] The economy without adjustment costs corresponds to $\phi = 0$.

The representative firm's objective is to maximize its pre-dividend stock market value, period by period, by choosing its investment, capacity utilization and labor input:

$$Max_{\{I_{t+j}^f, U_{t+j}^f, N_{t+j}^f\}}\left(D_t^f + P_t^e\right)$$

subject to:

$$\begin{aligned}
P_t^e &= E\left(\sum_{j=1}^{\infty} MRS_{t,t+j}^h D_{t+j}^f\right) + e^{\nu_t} \\
K_{t+j}^f &= \left(1 - \Omega_{t+j-1}^f\right) K_{t+j-1}^f + I_{t+j-1}^f \\
\Omega_{t+j}^f &= \tau\left(U_{t+j}^f\right)^\theta \\
Y_{t+j}^f &= A_{t+j} X_{t+j}\left(U_{t+j}^f K_{t+j}^f\right)^\alpha \left(N_{t+j}^f\right)^{1-\alpha} \\
D_{t+j}^f &= Y_{t+j}^f - W_{t+j} N_{t+j}^f - I_{t+j}^f - g_{t+j}\left(I_{t+j}^f - \Omega_{t+j}^f K_{t+j}^f\right)
\end{aligned} \tag{13}$$

The first order conditions for the firm's problem (13) determining its labor hiring decision, its capacity utilization choice, and its investment decision are:

$$(1-\alpha)\frac{Y_t^f}{N_t^f} = W_t, \tag{14}$$

[10] We also experimented with an asymmetric adjustment costs function introduced by Pfann and Verspagen [23] , where the adjustment cost in period t is given by: $g_t\left(I_t - \tau U_t^\theta K_t\right) = \frac{\phi}{2K_t}\left(I_t - \tau U_t^\theta K_t\right)^2 - \frac{\varphi}{K_t}\left(I_t - \tau U_t^\theta K_t\right) + \exp\left(\frac{\varphi}{K_t}\left(I_t - \tau U_t^\theta K_t\right)\right) - 1$. Quantitative results were essentially identical for symmetric and asymmetric costs.

$$\alpha \frac{Y_t^f}{U_t^f} = \tau \theta K_t^f \left(U_t^f\right)^{\theta-1}, \tag{15}$$

and

$$1 + \frac{\partial g_t(\cdot)}{\partial I_t^f} = E_t[MRS_{t,t+1}^h \tag{16}$$
$$\times \left\{ \frac{\alpha Y_{t+1}}{K_{t+1}^f} - \frac{\partial g_{t+1}(\cdot)}{\partial K_{t+1}^f} + 1 - \tau \left(U_{t+1}^f\right)^\theta + \frac{\partial g_{t+1}(\cdot)}{\partial I_{t+1}^f} - \tau \left(U_{t+1}^f\right)^\theta \frac{\partial g_{t+1}(\cdot)}{\partial I_{t+1}^f} \right\}] + e^{\nu_t}.$$

If adjustment costs are zero, (16) simplifies to:

$$-1 + E_t \left\{ MRS_{t,t+1}^h \left[\alpha \frac{Y_{t+1}^f}{K_{t+1}^f} + 1 - \tau \left(U_{t+1}^f\right)^\theta \right] \right\} + e^{\nu_t} = 0 \tag{17}$$

Since the private technology of a representative firm is convex, an interior solution to the model exists and the equilibrium is well-defined.[11]

2.3 Equilibrium

An equilibrium in this economy is a wage function $W_t = W(K_t, C_t, \epsilon_t)$, a share price function $P_t^e = P^e(K_t, C_t, \epsilon_t)$ and a risk-free security price function $P_t^b = P^b(K_t, C_t, \epsilon_t)$ such that (6), (7), (8), (14), (15), and (16) are satisfied, along with the market clearing conditions:

[11] If $\eta = 0$, capacity utilization has no effect on the homogeneity of the production function. To see this, substitute the optimal capacity from equation (15) into the production function to obtain the reduced form production function. The exponents on capital and labor sum to 1.

$$
\begin{aligned}
\int_0^1 N_t^f d\iota &= \int_0^1 N_t^h d\kappa = N_t^h = N_t \\
\int_0^1 K_t^f d\iota &= K_t \\
\int_0^1 Z_t^{ef} d\iota &= \int_0^1 Z_t^e d\kappa = Z_t^e = 1 \\
Z_t^b &= \int_0^1 Z_t^b d\kappa = 0 \\
C_t^h &= \int_0^1 C_t^h d\kappa = C_t \\
Y_t &= C_t + I_t + g_t \left(I_t - \tau U_t^\theta K_t \right)
\end{aligned} \tag{18}
$$

where κ represents the measure of households and ι is the measure of firms. The vector ϵ_t ($\epsilon_t \equiv [\nu_t, a_t]$) contains exogenous state variables: the sunspot shock and the technology shock ($a_t = \ln A_t$). When the externality is positive, the decentralized equilibrium in this economy is not Pareto optimal because firms fail to take the external effect into account when deciding on their factors of production.

Using the equilibrium conditions (18) and first order conditions of both the representative household and the firm, the model in Section 2 can be

equivalently represented as:

$$
\begin{aligned}
\alpha\frac{Y_t}{U_t} &= \tau\theta U_t^{\theta-1}K_t \\
MU_t &= \frac{\Lambda}{(1-\alpha)}\frac{N_t}{Y_t} \\
Y_t &= C_t + I_t + g_t\left(I_t - \tau U_t^{\theta}K_t\right) \\
Y_t &= A_t K_t^{\alpha(1+\eta)} U_t^{\alpha(1+\eta)} N_t^{(1-\alpha)(1+\eta)} \\
D_t &= Y_t - W_t N_t - I_t - g_t\left(I_t - \tau U_t^{\theta}K_t\right) \\
W_t &= (1-\alpha)\frac{Y_t}{N_t} \\
K_{t+1} &= \left(1-\tau U_t^{\theta}\right)K_t + I_t \\
1 + \frac{\partial g_t(\cdot)}{\partial I_t} &= E_t[MRS_{t,t+1}^h\left\{\frac{\alpha Y_{t+1}}{K_{t+1}} - \frac{\partial g_{t+1}(\cdot)}{\partial K_{t+1}} + 1 - \tau U_{t+1}^{\theta} + \frac{\partial g_{t+1}(\cdot)}{\partial I_{t+1}} - \tau U_{t+1}^{\theta}\frac{\partial g_{t+1}(\cdot)}{\partial I_{t+1}}\right\}] + e^{\nu_t} \\
sr_{t+1} &= \lambda sr_t + \left(\frac{1}{\overline{SR}} - 1\right)(c_{t+1} - c_t) \\
\ln A_t &= (1-\rho)\ln\overline{A} + \rho\ln A_{t-1} + \varepsilon_t
\end{aligned}
\tag{19}
$$

The first equation determines the optimal rate of capital utilization, the second describes equilibrium in the labor market, the third equation is a market clearing condition in the goods market, and the fourth is the aggregate (social) production function. The next two equations determine the aggregate dividend and the equilibrium wage rate. The four inter-temporal equations are the equation describing the evolution of the aggregate capital stock, the consumption Euler equation and two equations describing the evolution of habit and the process for the level of technology. System (19) does not have an analytical solution. To characterize the model dynamics, we solve (19) for the unique steady state implied by the above equilibrium conditions and then log-linearize around the steady state values. In this solution method we follow Campbell [7] and Uhlig [26]. The details of the log-linearization procedure are presented in Appendix A.

The state of the economy at time t is fully described by the vector $s_t \equiv [k_t, c_t, sr_t]$ of the endogenous state variables and by the exogenous

shocks, technology and sunspot, represented by the vector ϵ_t ($\epsilon_t \equiv [\nu_t, a_t]$). We collect endogenous control variables (output, employment, investment, capacity utilization, dividend and wage rate) in the vector e_t. A solution to the log-linearized system and therefore a stationary rational expectations equilibrium within the approximate economy is given by the following laws of motion:

$$\begin{aligned} s_t &= P s_{t-1} + Q\epsilon_t \\ e_t &= R s_{t-1} + S\epsilon_t. \end{aligned} \tag{20}$$

If both eigenvalues of the matrix P in (20) are stable (i.e. less than 1 in absolute value), the economy exhibits indeterminacy and the steady state is therefore a sink.

2.4 Rates of Return

We define the one period return on equity as:

$$R^e_{t,t+1}\left(s_t, \epsilon_t\right) = \frac{P^e_{t+1}(s_{t+1}, \epsilon_{t+1}) + D_{t+1}(s_{t+1}, \epsilon_{t+1})}{P^e_t\left(s_t, \epsilon_t\right)} \tag{21}$$

The risk-free security in our economy is an asset paying one unit of the consumption good in every state next period. This asset is in zero net supply. The conditional risk-free rate is given by:

$$R^b_{t,t+1}\left(s_t, \epsilon_t\right) = \frac{1}{P^b_t\left(s_t, \epsilon_t\right)} \tag{22}$$

Technology and belief shocks in the linearized economy (20) are drawn from the normal distribution. Hence, the probabilistic structure of the model accommodates the log-linear, log-normal pricing formulae extensively used by Jermann [17] and Avalos [1]. Using the log-linear, log-normal framework, we derive closed-form expressions for the financial returns, which highlight relationships between returns on financial assets and real macroeconomic variables in the model.

At time t, a claim to a single uncertain normalized future payout D_{t+j}

is priced by its expected present discounted value:

$$P_t\left(D_{t+j}\right) = E\left[MRS_{t,t+j} D_{t+j}\right] \tag{23}$$

Using the definition of the risk-free rate (22), the price of the risk-free asset is found as a special case of the Euler equation (23):

$$\begin{aligned} R^b_{t,t+1} &= \frac{1}{P^b_t\left(1_{t+1}\right)} \\ &= e^{\left\{-E_t[mrs_{t,t+1}]-\frac{1}{2}Var_t[mrs_{t,t+1}]\right\}} \end{aligned} \tag{24}$$

Using equation (21) , the price of the equity security is calculated as the price of a claim to the infinite sequence of dividends $\{D_{t+j}\}_{j=1}^{\infty}$, with each dividend payment defined by equation (12) and priced according to (23):

$$\begin{aligned} R^e_{t,t+1}\left(\{D_{t+j}\}_{j=1}^{\infty}\right) &= \frac{P^e_{t+1}\left(\{D_{t+j}\}_{j=2}^{\infty}\right) + D_{t+1}}{P^e_t\left(\{D_{t+j}\}_{j=1}^{\infty}\right)} \\ &= \frac{\sum_{j=1}^{\infty} E_{t+1}\left[e^{mrs_{t,t+j}+d_{t+j+1}}\right] + e^{d_{t+1}}}{\sum_{j=1}^{\infty} E_t\left[e^{mrs_{t,t+j}+d_{t+j}}\right]} \end{aligned} \tag{25}$$

The derivation of returns on financial securities and their unconditional moments is relegated to Appendix B. We are able to obtain closed-form expressions for the unconditional expectation (26) and standard deviation (27) of the risk-free rate. We analytically compute period by period equity returns according to equation (28) and use simulations to find their first and second unconditional moments.

3 Results

We now present quantitative results for several formulations of the model described in Section 2. The main objective of this research is to study the behavior of financial assets in the context of local indeterminacy while ensur-

ing that the models under consideration adequately summarize key business cycle facts. Our benchmark specification is the model with variable capacity utilization and aggregate increasing returns to scale in production (IRS). In Section 3.4 we incorporate modifications such as habit formation (HF), costs of capital adjustments (COA) and their combination into the benchmark economy with IRS. We also examine the corresponding economies with constant returns to scale in production, indexed by CRS. We simulate each version and examine each set of business cycle characteristics and financial properties. Overall, we report results from eight specifications.

3.1 Parameter Choices

The parameter choices for each considered variant of the model are presented in Tables 1 and 2. In Table 1 we display parameter values common to all specifications. They are standard in the RBC literature (see for example Hansen [13], Mehra and Prescott [22] and King and Rebelo [19]).

Table 1: Parameter Choices

α	β	ξ	θ
0.3	0.99	3	1.4

Capital's share of output, α, is 0.3. The subjective discount factor, β, is 0.99, corresponding to a steady state risk-free rate of return of 4% per year. Thus we simulate the model at the quarterly frequency. The coefficient of relative risk aversion, ξ, is equal to 3, which is considered conservative in the asset pricing literature. Below we also present results using a higher value. Following Wen, we set $\theta = 1.4$ and $\tau = 1/\theta$, implying a steady state depreciation rate of 2.5% per quarter.

Table 2 displays the values of the parameters that change across specifications of the model. In the benchmark specification equilirium ceases to be unique when the externality parameter exceeds 0.1036 (see Wen [27]). In the benchmark parameterization, we choose $\eta = 0.15$. Given recent evidence

Table 2: Other Parameter Choices

Parameter:	η	λ	$\overline{SR}$	ϕ	shock (std. deviation percent per quarter)
Benchmark	0.15	–	–	0	ν (.054)
IRS HF	0.15	0.9658	0.0702	0	ν (0.0525)
IRS COA	0.15	–	–	0.2	ν (0.034)
IRS HFCOA	0.15	0.9658	0.0702	0.2	ν (0.033)
CRS	0	–	–	0	ε (0.315)
CRS HF	0	0.9658	0.0702	0	ε (0.377)
CRS COA	0	–	–	0.2	ε (0.362)
CRS HFCOA	0	0.9658	0.0702	0.2	ε (0.413)

in Basu and Fernald [2] this falls within an empirically plausible range for the level of aggregate increasing returns. We considered the following two effects when choosing the value of η. First, a larger value leads to a longer cycle because with higher returns to scale the advantages of keeping output high are greater, leading to longer booms. At the same time it is easy to get stuck in a recession as falls in inputs lead to large and persistent declines in output. At $\eta = 0.15$ the cycle length is 12 years.[12] Second, a larger externality leads to more variable consumption because with a higher externality the returns to investment are higher and agents therefore find it less advantageous to smooth consumption. To gain a more detailed understanding of the model's responses to the change in the magnitude of the externality parameter, below we present results for a higher value of η. There are no technology shocks in the benchmark specification, so that $\epsilon \equiv \nu$.

In the economies where agents' preferences include habit formation (HF), we use quarterly versions of Campbell and Cochrane's [9] values for the habit persistence parameter λ and the steady state surplus ratio $\overline{SR}$ in equation (4). Empirical work on investment functions in the U.S. data provides little help in calibrating parameters of the cost of adjustment. In the economies with capital adjustment costs (COA) we fix ϕ at 0.2.

In CRS specifications, where the externality parameter is $\eta = 0$ the

[12] As another example, with $\eta = 0.2$ the length of the cycle is 64 years.

economies are not susceptible to sunspot shocks. We use a technology shock with $\rho = 0.95$ as a source of exogenous uncertainty. In each formulation, we choose the standard deviation of the exogenous shock to match the standard deviation of simulated quarterly HP-filtered output in the U.S. data (1.82%).

3.2 Critical Values of the Externality

In Section 3.4, we add habit formation and adjustment costs to the benchmark economy. We focus on asset pricing exploration in the presence of multiple equilibria and sunspot shocks and compare the results obtained from different specifications of the studied model. It is therefore important to verify that in each modified model indeterminacy still results with empirically plausible returns to scale, and in particular at the chosen level of increasing returns ($\eta = 0.15$). Table 3 shows the critical value of increasing returns (η) in each formulation of the model, after which the equilibrium is not unique but locally indeterminate.

Table 3: Critical Values of Externality

Model:	Critical Value of Externality, η
Benchmark	0.1036
IRS HF	0.1003
IRS COA	0.1119
IRS HFCOA	0.1189

It turns out that neither habit formation nor adjustment costs have serious effect on the threshold externality value. Our result differs from conclusions in Kim [18] and Wen [28] who introduce adjustment costs into the one-sector production model with increasing returns to scale without variable capacity utilization and find that the model has a unique equilibrium for any realistic value of the externality parameter. Herrendorf and Valentinyi [16] consider COA in an economy with sector-specific externalities and come to the same conclusion. We attribute the difference to the effect of variable capacity utilization. Acting as an additional factor in the

production function (9), variable capacity undermines the rigidity of the capital stock brought about by costly capital adjustments, allowing agents' expectations to be self-fulfilled.

3.3 Stylized Business Cycle and Financial Facts: the Benchmark Economy

Table 4 summarizes the macroeconomic and financial behavior of the benchmark economy. For purposes of comparison, column one shows the statistics derived from U.S. data and column two from Hansen's [13] indivisible labor model, driven by persistent technology shocks, a standard in the real business cycle literature. Column three reports the results from the benchmark specification. In columns four and five, we vary risk aversion and the size of the externality respectively.

In the benchmark case, the standard deviations of the capital stock and employment closely match those in the data, improving on the results of the standard RBC model. The volatility of investment is higher and the volatility of the real wage lower than in the data. Although no data are available for the aggregate U.S. economy's capacity utilization rate, data from the manufacturing sector indicate that capacity utilization is about as variable as output (and is procyclical), which is what the benchmark model predicts. A serious drawback of the model, with significant implications for asset pricing, is its extremely smooth consumption. The very low standard deviation of consumption is a direct result of the variable capacity utilization, which provides an extra margin for consumption smoothing in addition to the investment and labor choices. With respect to correlations with output, the benchmark model does about as well as the RBC model, although consumption's correlation is understated.

Now we turn our attention to the financial statistics. The benchmark model performs marginally better in this regard than the standard RBC model but still fails to replicate the moments of financial returns found in the U.S. data. The equity premium ($\pi = E[r_e] - E[r_b]$) produced by the model, 0.09%, is far below the 7.2% observed in the data. The model's 4.1% risk-

Table 4: Macroeconomic and Financial Statistics for the Benchmark model

	U.S. data	RBC Model	Benchmark	High risk aversion	High externality
ξ		3	3	10	3
η		0	0.15	0.15	0.2
Macroeconomic Statistics: Standard Deviations					
σ_y	1.82	1.82	1.82	1.91	0.78
σ_c	1.35	0.69	0.05	0.02	0.03
$\frac{\sigma_c}{\sigma_y}$	0.74	0.38	0.03	0.0085	0.042
σ_n	1.79	0.95	1.71	1.80	0.69
σ_k	0.63	0.47	0.80	0.93	0.38
σ_i	5.3	5.82	8.35	8.89	3.51
σ_u	N/A	N/A	1.49	1.57	0.63
σ_w	0.68	0.93	0.15	0.16	0.10
$\sigma_{\Delta div}$	7.2		9.25	9.37	4.50
Macroeconomic Statistics: Correlations with Output					
$\rho_{c,y}$	0.88	0.86	0.73	0.71	0.88
$\rho_{n,y}$	0.88	0.98	1	1	1
$\rho_{k,y}$	0.04	0.07	-0.07	-0.09	-0.07
$\rho_{i,y}$	0.8	0.99	1	1	1
$\rho_{u,y}$	N/A	N/A	0.91	0.91	0.91
$\rho_{w,y}$	0.12	0.98	0.73	0.71	0.88
Financial Statistics					
$E[r_e]$	8.1	4.11	4.19	4.18	4.18
σ_{r_e}	15.6	0.53	0.86	0.85	0.44
$E[r_b]$	0.9	4.11	4.10	4.10	4.10
σ_{r_b}	1.7	0.35	0.023	0.023	0.01
π	7.2	0.0006	0.09	0.08	0.08

Business cycle statistics for the U.S. economy are from King and Rebelo [19], whose data set is from Stock and Watson [25] and covers the period from 1947 (first quarter) to 1996 (fourth quarter). Data sources are described in Stock and Watson. All numbers are given in percents. Statistics for the macroeconomic variables in the U.S. data and in artificial economies are per quarter and calculated for the logs of variables detrended with the HP filter. Financial statistics for the U.S. economy are from Mehra and Prescott [22] and Campbell [8]. Financial returns and their standard deviations are annualized.

free rate is high relative to its empirical counterpart. The volatility puzzle is also present in our model. The equity return is a little more volatile than in the standard RBC model, but the standard deviation of the risk-free rate is very low. In the benchmark setting, the representative agent has plenty of flexibility in his response to shocks. For this reason, his expectations are fulfilled at a lower, and hence more realistic, level of increasing returns than in the comparable economy without the variable capacity utilization. Variable capacity, however, leads to extremely smooth consumption and low standard deviations of the marginal rate of substitution in consumption and the stochastic discount factor, which is detrimental to asset pricing. The different nature of shocks and different propagation mechanism of these shocks in the model economy do not sufficiently improve the asset pricing results.

The last two columns in Table 4 report statistics from the benchmark model for different values of the parameters ξ and η respectively. As expected, the increase in the coefficient of relative risk aversion lowers the standard deviation of consumption even further. The volatility of all other macro variables therefore increases, because more risk-averse agents vary investment, capacity utilization, hours worked and, as a result, the capital stock and output to avoid fluctuations in their consumption. The change in risk aversion leaves correlations largely unaffected. On the financial side, the results are robust to the change in the risk-aversion parameter.

The increase in the externality parameter, η, lowers the overall volatility of the economy. With higher increasing returns to scale, agents do not have to vary factors of production as much in response to an expectational change to ensure that their beliefs are fulfilled. In addition, less trading in financial assets takes place, which explains the lower second moments of their returns.

We can summarize the performance of the benchmark model as follows: like the standard RBC model, it adequately represents the stylized business cycle statistics but does not resolve financial puzzles.

3.4 Modifications to the Benchmark Economy

Several features, such as habit persistence and capital adjustment costs, have been successfully introduced into RBC models to improve their ability to reproduce stylized financial market facts. The question arises as to whether these modifications will similarly improve asset pricing results in the benchmark model without disturbing its success along business cycle dimensions. Table 5 contains the relevant macroeconomic statistics.

The inclusion of habit formation into the benchmark specification (IRS HF) leaves macroeconomic statistics virtually unaffected, with the notable exception of consumption, whose standard deviation decreases almost 15-fold. Habit drives a wedge between the agent's risk aversion and his/her degree of inter-temporal substitution. The very low elasticity of intertemporal substitution implies that agents are extremely reluctant to vary their consumption streams through time. The addition of capital adjustment costs (IRS COA and IRS HFCOA) does not change the macroeconomic summary of the model relative to the specifications without these costs.

In Table 6 we record corresponding financial statistics. The inclusion of habit (IRS HF), adjustment costs (IRS COA) and their combination (IRS HFCOA) does not change the risk-free rate and has only a marginal effect on the equity return. These modifications do not alleviate the volatility puzzle either. While it was shown in previous research (for example Jermann [17] and Avalos [1]) that the addition of habit persistence or adjustment costs individually does not improve the financial performance in RBC models, their combination has proven to be successful in solving the equity premium puzzle and increasing the standard deviation of the return on equity. In our setting with variable capacity utilization, there is no such improvement because even intertemporally very risk-averse habit-forming agents are able to smooth their consumptions streams to a desired degree by choosing how much they wish to work and utilize capital. Adjustment costs make changes in investment costly, but do not prevent agents from utilizing capital stock at different rates. Consequently, there is no need to increase the return on equity and to reduce the return on the risk-free security to make the former

Table 5: Macroeconomic Summary of Models with Variable Capacity Utilization

Model	Standard Deviations							
	σ_y	σ_c	σ_n	σ_k	σ_i	σ_u	σ_w	$\sigma_{\Delta div}$
U. S. data	1.82	1.35	1.79	0.63	5.3	N/A	0.68	7.2
Benchmark	1.82	0.05	1.71	0.80	8.35	1.49	0.15	9.25
IRS HF	1.82	0.0036	1.74	0.90	8.60	1.52	0.15	9.23
IRS COA	1.82	0.05	1.74	0.78	8.46	1.48	0.14	8.59
IRS HFCOA	1.82	0.0033	1.73	0.77	8.55	1.48	0.14	8.70
CRS	1.82	0.13	1.53	0.53	8.21	1.30	0.38	11.91
CRS HF	1.82	0.0042	1.40	0.57	8.53	1.31	0.50	12.60
CRS COA	1.82	0.15	1.39	0.52	8.13	1.26	0.46	11.29
CRS HFCOA	1.82	0.0041	1.34	0.57	8.54	1.31	0.56	12.39
	Contemporaneous Correlations with Output							
		ρ_{cy}	ρ_{ny}	ρ_{ky}	ρ_{iy}	ρ_{uy}	ρ_{wy}	
U. S. data		0.88	0.88	0.04	0.8	N/A	0.12	
Benchmark		0.73	1	-0.07	1	0.91	0.73	
IRS HF		0.58	1	-0.10	1	0.91	0.71	
IRS COA		0.76	1	-0.10	1	0.93	0.76	
IRS HFCOA		0.65	1	-0.11	1	0.93	0.76	
CRS		0.86	0.99	0.20	1	0.96	0.86	
CRS HF		-0.99	0.99	0.16	1	0.95	0.88	
CRS COA		0.91	0.99	0.19	1	0.96	0.91	
CRS HFCOA		-0.99	0.98	0.15	1	0.95	0.90	

Business cycle statistics for the U.S. economy are from King and Rebelo [19], whose data set is from Stock and Watson [25] and covers the period from 1947 (first quarter) to 1996 (fourth quarter). Data sources are described in Stock and Watson. All numbers are given in percents. Statistics for the macroeconomic variables in the U.S. data and in artificial economies are per quarter and calculated for the logs of variables detrended with the HP filter. In all artificial economies with increasing returns to scale (IRS) $\eta = 0.15$.

Table 6: Financial Summary of the Models with Variable Capacity Utilization

Model	Financial Statistics				
	$E[r_e]$	σ_{r^e}	$E[r_b]$	σ_{r^b}	π
U. S. data	8.10	15.60	0.90	1.70	7.20
Benchmark	4.19	0.86	4.10	0.023	0.09
IRS HF	4.19	0.83	4.10	0.022	0.09
IRS COA	4.18	0.40	4.10	0.017	0.08
IRS HFCOA	4.18	0.39	4.10	0.016	0.08
CRS	4.17	0.22	4.10	0.037	0.07
CRS HF	4.17	0.26	4.10	0.025	0.07
CRS COA	4.17	0.25	4.10	0.018	0.07
CRS HFCOA	4.17	0.29	4.10	0.016	0.07

Financial statistics for the U.S. economy are from Mehra and Prescott [22] and Campbell [8]. Financial returns and their standard deviations are annualized. In all artificial economies with increasing returns to scale (IRS) $\eta = 0.15$.

relatively more attractive than the latter. Again, the elements of the model that lead to indeterminacy at reasonable returns to scale undermine the influence of factors that would enable the model to match the asset pricing facts.

Up until now, we have identified variable capacity utilization as a main reason for failures to explain the stylized financial facts in the studied economies. But these economies also differ from the standard RBC model because of their increasing returns to scale in production and use of sunspot shocks as the source of fluctuations. In order to determine whether these features have a significant impact on the macroeconomic and financial market performance of models under investigation, we obtain results from economies analogous to those already discussed, but with constant returns to scale. The equilibrium is unique in these economies; and fluctuations are driven by technology shocks. Pairwise comparison of corresponding IRS and

CRS specifications in Tables 5 and 6 reveals that in each case the volatilities of consumption, the real wage and the growth rate of dividends are slightly higher in the CRS setting. Wen [27] points out that the elasticity of the reduced form labor demand curve, evaluated at the optimal rate of capacity utilization, is given by $\frac{(1-\alpha)(1+\eta)\theta}{\theta-\alpha(1+\eta)} - 1$. In the IRS cases, with $\eta = 0.15$ this elasticity is close to zero (equal to 0.068). Inelastic demand for labor combined with the flat supply curve implies almost constant real wage and consumption. In the CRS cases the slope of labor demand is further away from zero (it is equal to -0.11) and therefore real wages and consumption are slightly more variable. Since we fix the standard deviation of output at 1.82%, more variable consumption results in less variable investment. Most notably, the financial statistics are very similar in the corresponding CRS and IRS specifications.

4 Conclusion

We set out to examine the pricing of financial assets in dynamic production economies with indeterminacy of equilibria and sunspot shocks. We wished to focus on models, which adequately present stylized business cycle facts and do not require unrealistically high increasing returns in production for expectations to become self-fulfilling. Our production economies with variable utilization of the capital stock fit both of the above criteria. We have found that the studied models marginally increase the equity premium relative to the traditional RBC models, but this improvement is insufficient to deem them a success. When compared to the U.S. financial market data, the risk premium and volatility of financial assets are low.

In an effort to improve the asset pricing implications of the benchmark model, we have modified our original specification to include habit persistence, adjustment costs and their combination. Our results indicate that in contrast to standard RBC models with unique equilibria, these features do not solve the financial puzzles in the benchmark model.

We explain the poor financial performance of the studied economies by inclusion of the variable utilization of capital, which provides risk-averse

agents with an additional channel for consumption smoothing in response to shocks. On the positive side, the increased flexibility in responses to shocks allows agents' expectations to be fulfilled at levels of increasing returns low enough to be justified empirically. On the negative side, the variable capacity utilization is powerful enough to undermine any rigidities introduced into the model economy to improve its asset pricing performance. It is well known that in the RBC models features which pose obstacles to consumption smoothing by risk-averse agents (for example adjustment costs) are necessary to obtain good asset pricing results. Our conclusion is that it is not possible to achieve satisfactory resolution of the asset pricing puzzles in an economy with local indeterminacy obtained with an empirically plausible production externality. The different nature of shocks and the alternative propagation mechanism do not seem to have a significant impact on the financial performance of the economies under investigation.

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A Solving for the Equilibrium Law of Motion

In indeterminacy cases the recursive competitive equilibrium conditions are log-linearized around the non-stochastic steady state and ordered as follows:

$$
\begin{aligned}
0 &= A_1 s_t + A_2 e_t \\
0 &= B_1 s_{t+1} + B_2 e_{t+1} + B_3 s_t + B_4 e_t + B_5 \epsilon_{t+1}
\end{aligned}
$$

where $s_t = [k_t, c_t, sr_t]$ is a vector of endogenous state variables. If habit formation is not included into the representative agent's preferences, the state vector is $s_t = [k_t, c_t]$. e_t is a vector of other endogenous variables and ϵ_t is a vector of exogenous shocks. The stationary rational expectations equilibrium is given by (20) with

$$
\begin{aligned}
P &= -\left(B_1 - B_2 A_2^{-1} A_1\right)^{-1} \left(B_3 - B_4 A_2^{-1} A_1\right) \\
Q &= -\left(B_1 - B_2 A_2^{-1} A_1\right)^{-1} B_5 \\
R &= -A_2^{-1} A_1 P \\
S &= -A_2^{-1} A_1 Q
\end{aligned}
$$

In determinacy cases we use the log-liner approximation and the method of undetermined coefficients following Uhlig [26]. The state vector is given by $s_t = [k_t, sr_t]$ or in the absence of habit formation by $s_t = [k_t]$. The equations are ordered as follows:

$$
\begin{aligned}
0 &= C_1 s_t + C_2 s_{t-1} + D e_t + U \epsilon_t \\
0 &= E_t \left[F s_{t+1} + G s_t + H s_{t-1} + J e_{t+1} + K e_t + L \epsilon_{t+1} + M \epsilon_t \right] \\
\epsilon_{t+1} &= N \epsilon_t + \varsigma_{t+1}; E_t \left[\varsigma_{t+1} \right] = 0
\end{aligned}
$$

where ς_t is a vector of random impulses $(\varsigma_t = [\nu_t, \varepsilon_t])$.

B Derivation of Unconditional Moments of Returns on Financial Securities in Log-linear Log-normal Environment

We combine the state variables in the vector $x_t = [s'_{t-1}, \epsilon'_{t-1}]$ and express dynamics of the system describing the recursive equilibrium law of motion as a first order VAR with normal i.i.d. impulses as follows:

$$x_t = M x_{t-1} + W \varsigma_t$$

where $M = \begin{bmatrix} P & QN \\ 0 & N \end{bmatrix}$ and $W = \begin{bmatrix} Q \\ I \end{bmatrix}$.

We also rewrite the vector of endogenous variables e_t as a linear combination of the state vector x_{t-1} and the impulse ς_t :

$$e_t = T x_{t-1} + S \varsigma_t$$

where $T = \begin{bmatrix} R & SN \end{bmatrix}$

B.1 Derivation of the Unconditional Expectation and Variance of the Risk-Free Rate

Risk-free rate is a return on a one-period bond with a sure payout of one unit of the consumption good. The price of the riskless bond is given by:

$$
\begin{aligned}
P_t^b\left(1_{t+1}\right) &= E_t\left[MRS_{t,t+1}1_{t+1}\right] \\
&= \beta e^{\{E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]+\frac{1}{2}Var_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]\}}
\end{aligned}
$$

The above expression is derived using a formula for the expectation of a log-normal variable.

The risk-free rate of return is:

$$
\begin{aligned}
R_{t,t+1}^b &= \frac{1}{P_t^b\left(1_{t+1}\right)} \\
&= \beta^{-1} e^{\{-E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]-\frac{1}{2}Var_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]\}}
\end{aligned}
$$

Utilizing the fact that conditional variances are constant and using the law of iterated expectations, we can derive an unconditional expectation of the risk-free rate:

$$
\begin{aligned}
E\left[R_{t,t+1}^b\right] &= \beta^{-1} e^{\{-E[E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]+\frac{1}{2}Var_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]]+\frac{1}{2}Var[E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]]\}} \\
&= \beta^{-1} e^{\{\frac{1}{2}Var[E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]]-\frac{1}{2}Var[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})-E_t[\xi(c_t+sr_t-c_{t+1}-sr_{t+1})]]\}}
\end{aligned}
\tag{26}
$$

Both variances in the above expression have closed form solutions as they are functions of consumption c_t and sr_t (if habit is present).

In the indeterminacy case both of these variables are elements of the state vector x_t. Let M_c, M_s, W_c and W_s be row vectors of the matrices M and W respectively that correspond to the consumption and surplus ratio

variables. The variance terms are:

$$
\begin{aligned}
&Var[E_t[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)]] \\
&= \xi^2 \left(M_c + M_s\right)\left(M - I\right) Var\left(x_{t-1}\right)\left(M - I\right)'\left(M_c + M_s\right)' \\
&+ \xi^2 \left(\left(M_c + M_s\right) W - W_c - W_s\right) \Sigma \left(\left(M_c + M_s\right) W - W_c - W_s\right)'
\end{aligned}
$$

and

$$
\begin{aligned}
&Var[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right) - E_t[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)]] \\
&= \xi^2 \left(W_c + W_s\right) \Sigma \left(W_c + W_s\right)'
\end{aligned}
$$

In the determinacy case, c_t is a element of the vector e_t. Let T_c and S_c be the row vectors of matrices T and S, corresponding to consumption. Then the variance terms in equation (26) are given by:

$$
\begin{aligned}
&Var[E_t[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)]] \\
&= \xi^2 \left(T_c + M_s\right)\left(M - I\right) Var\left(x_{t-1}\right)\left(M - I\right)'\left(T_c + M_s\right)' \\
&+ \xi^2 \left(\left(T_c + M_s\right) W - S_c - W_s\right) \Sigma \left(\left(T_c + M_s\right) W - S_c - W_s\right)'
\end{aligned}
$$

and

$$
\begin{aligned}
&Var[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right) - E_t[\xi\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)]] \\
&= \xi^2 \left(S_c + W_s\right) \Sigma \left(S_c + W_s\right)'
\end{aligned}
$$

To calculate the unconditional variance of the risk-free rate, we use the formula for the variance of log-normal distribution. Suppose

$$
\log\left(Z\right) \sim N\left(\mu, \sigma^2\right)
$$

then the variance of log-normal variable Z is given by:

$$Var\left(Z\right) = e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$$

Using the equation (22) we compute the unconditional expectation of $\log\left(R^b_{t,t+1}\right)$

$$E\left[\log\left(R^b_{t,t+1}\right)\right] = -\log\beta - \frac{1}{2}\xi^2 E[Var_t\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)]$$

The unconditional variance of $\log\left(R^b_{t,t+1}\right)$ is equal to:

$$Var\left(\log\left(R^b_{t,t+1}\right)\right) = Var\left(E_t\left(c_t + sr_t - c_{t+1} - sr_{t+1}\right)\right)$$

$$\begin{aligned} Var(R^b_{t,t+1}) = \beta^{-2} e^{-\xi^2 E[Var_t(c_t+sr_t-c_{t+1}-sr_{t+1})]+Var(E_t(c_t+sr_t-c_{t+1}-sr_{t+1}))} \\ \times\left(e^{Var(E_t(c_t+sr_t-c_{t+1}-sr_{t+1}))}-1\right) \end{aligned} \tag{27}$$

B.2 Unconditional Expectation and Variance of the Rate of Return on a Stock

Stocks constitute a claim to an infinite sequence of uncertain dividends paid off by a firm. Consequently, they can be priced as a collection of single future payoffs:

$$\begin{aligned} P^e_t\left(\{D_{t+j}\}_{j=1}^{\infty}\right) &= \sum_{j=1}^{\infty} P^e_t\left(D_{t+j}\right) = \sum_{j=1}^{\infty} E_t\left[MRS_{t,t+j} D_{t+j}\right] \\ &= \sum_{j=1}^{\infty} \beta^j E_t\left[e^{\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j}}\right] \end{aligned}$$

The one period return on the stock is:

$$R^e_{t,t+1}\left(\{D_{t+j}\}_{j=1}^{\infty}\right) = \frac{P^e_{t+1}\left(\{D_{t+j}\}_{j=2}^{\infty}\right) + D_{t+1}}{P^e_t\left(\{D_{t+j}\}_{j=1}^{\infty}\right)} \tag{28}$$
$$= \frac{\sum_{j=1}^{\infty}\beta^j E_{t+1}\left[e^{\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j+1}}\right] + e^{d_{t+1}}}{\sum_{j=1}^{\infty}\beta^j E_t\left[e^{\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j}}\right]}$$

Stock returns can be computed from the model linear solution, but we have to simulate the model to find their unconditional moments. We will use an expectation property of log-normal distribution for simulations:

$$E_t\left[e^{\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j}}\right] = e^{E_t(\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j})+\frac{1}{2}Var_t(\xi(c_t+sr_t-c_{t+j}-sr_{t+j})+d_{t+j})} \tag{29}$$

In the indeterminacy case the expectation term in (29) is given by:

$$E_t\left(\xi\left(c_t+sr_t-c_{t+j}-sr_{t+j}\right)+d_{t+j}\right)$$
$$= \left[\xi\left(M_c+M_s\right)+\left(T_d-\xi\left(M_c+M_s\right)\right)M^j\right]x_{t-1}+\left[\xi\left(W_c+W_s\right)+\left(T_d-\xi\left(M_c+M_s\right)\right)M^{j-1}W\right]\varsigma_t$$

and the variance term is:

$$Var_t\left(\xi\left(c_t+sr_t-c_{t+j}-sr_{t+j}\right)+d_{t+j}\right)$$
$$= \sum_{s=1}^{j-2}\left(T_d-\xi\left(M_c+M_s\right)\right)M^sW\Sigma W'M^{s\prime}\left(T_d-\xi\left(M_c+M_s\right)\right)'$$
$$+\left(S_d-\xi\left(W_c+W_s\right)\right)\Sigma\left(S_d-\xi\left(W_c+W_s\right)\right)'$$

In the determinacy case the expectation and variance terms in (29) are as follows:

$$E_t\left(\xi\left(c_t+sr_t-c_{t+j}-sr_{t+j}\right)+d_{t+j}\right)$$
$$= \left[\xi\left(T_c+M_s\right)+\left(T_d-\xi\left(T_c+M_s\right)\right)M^j\right]x_{t-1}+\left[\xi\left(S_c+W_s\right)+\left(T_d-\xi\left(T_c+M_s\right)\right)M^{j-1}W\right]\varsigma_t$$

and

$$
\begin{aligned}
& Var_t\left(\xi\left(c_t + sr_t - c_{t+j} - sr_{t+j}\right) + d_{t+j}\right) \\
& = \sum_{s=1}^{j-2} \left(T_d - \xi\left(T_c + M_s\right)\right) M^s W \Sigma W' M^{s\prime} \left(T_d - \xi\left(T_c + M_s\right)\right)' \\
& + \left(S_d - \xi\left(S_c + W_s\right)\right)\Sigma\left(S_d - \xi\left(S_c + W_s\right)\right)'
\end{aligned}
$$

To calculate the unconditional expectation and variance of the return on a stock, we use the simulation results to calculate the series of prices and returns on equity. Then the mean and variance of the stock return is computed as usual:

$$
\begin{aligned}
E\left[R^e_{t,t+1}\right] &= \frac{1}{T}\sum_{t=1}^{T} R^e_{t,t+1} \\
Var\left[R^e_{t,t+1}\right] &= \frac{1}{T-1}\sum_{t=1}^{T}\left(R^e_{t,t+1} - E\left[R^e_{t,t+1}\right]\right)^2
\end{aligned}
$$