Tracing Externalities as Sources of Indeterminacy

Sharon Harrison*
Department of Economics
Barnard College
3009 Broadway
New York, NY 10027
U. S. A.
sh411@columbia.edu

Mark Weder
Department of Economics
Humboldt University Berlin
Spandauer Str.1
10178 Berlin
Germany
weder@wiwi.hu-berlin.de

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Abstract

This paper offers further inquiry into sources of indeterminacy in general equilibrium. We take a more agnostic approach than previous work as we allow for a more general interpretation of externalities. In particular, we identify the origin of scale economies as being from either labor or capital. We find that in the one-sector model it is the externalities associated with labor that generate the result. However, in the two-sector model, indeterminacy can be ultimately traced back to the externalities from capital in the investment sector.

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1 Introduction

Recent research in macroeconomics has focussed on models in which business cycles are driven by self-fulfilling changes in agents' beliefs[1]. In such models, the resulting possibility of a continuum of equilibria, or indeterminacy of equilibria, arises because of some market imperfection. This market imperfection may come from increasing returns to scale in production, often driven by external effects.

In this paper we study such a model. Our focus, however, is different than that of previous work in that we are interested in finding particular sources of the external effects. That is, we seek to determine more precisely what kind of external effect is operating in generating multiple equilibria. There has been some speculation about this in the indeterminacy literature. For example, Benhabib and Farmer (1994, p.19) propose that the externalities are "[...] spillover effects of knowledge acquisition". One is more likely, however, to find hypotheses about the sources of the externalities in empirical work estimating production functions. Bartlesman, Caballero and Lyons (1994) attempt to distinguish between externalities that come from suppliers and from customers. Caballero and Lyons (1992, p.211) say that "[...] the externalities are unlikely to be of the strict Marshallian type. Most likely they correspond to transaction or thick-market externalities arising from easier matching between agents during expansions." There is much skepticism, however, as to evidence that such explanations are based on reality. Basu and Fernald (1995, p.166) state: "It is difficult, however, to identify many examples of such thick-markets effects that affect specific manufacturing plants."

In this paper, we think of externalities in a different way. In particular,

[1]These include Benhabib and Farmer (1994, 1996, 1999), Benhabib and Nishimura (1998), Christiano and Harrison (1999), Farmer and Guo (1994), Harrison (1999) and Weder (1998, 1999).

we propose that they are *factor-specific.* That is, that economy-wide increases in capital and labor have *separate* beneficial effects on the production of individual firms. The capital externality can be thought of arising from shared infrastructure, or from strategic complementarities from industrialization. For example, Murphy, Shleifer and Vishny (1989, p.1005) claim that "[...] the profitability of investment depends on there being enough other sectors to industrialize so that high future spending justifies putting down a large-scale plant today. [...] coordinated investment across sectors leads to the expansion of markets for all industrial goods, and can thus be self-sustaining even when no firm can break even investing alone". Matsuyama (1991) has a similar argument in a model of international trade. The labor externality may reflect the idea that in booms workers exert more search effort, as in Howitt and McAfee (1987); or that as workers becomes more active, the optimal level of employment rises as costs of production fall, as in Blanchard and Summers (1988). Though our model differs from those used in the above papers, the intuition for the external effect carries over.

The innovative aspect of our work is that we incorporate these ideas about externalities into a theoretical model that obtains multiple equilibrium solutions. In particular, we study a less restrictive version of the continuous time two sector model of Benhabib and Farmer (1996). It is now well known that indeterminacy can result in this model with empirically plausible returns to scale. However, there is still some question as to whether a slight modification of the model can demonstrate that the requirements for indeterminacy are even less stringent. Therefore, in order to differentiate between necessary and superfluous effects, we allow externalities to be factor-specific. In addition, distinct externalities are derived from the use of sector-specific versus economy-wide capital and labor.

This paper is not the first to allow for asymmetric effects of capital and labor. Benhabib and Nishimura (1998) specify two and three sector models in which factor shares and externalities are allowed to differ across sectors. In addition, they allow for decreasing internal and constant overall returns to scale in production. Thus, increasing returns may come from fixed costs rather than from a declining marginal cost schedule. We specify only differing externalities, so as to isolate the importance of these effects while assuming constant returns at the firm level. Furthermore, Benhabib and Nishimura restrict their inquiry into the two-sector model to the special case of utility that is linear in consumption while we examine only logarithmic utility.[2]

In our set-up, depending on the restrictions we place on certain parameters, the model can collapse to only one sector, so that we have a more general version of the Benhabib-Farmer-Guo model. In this case, we find that both capital and labor externalities are needed for indeterminacy, but that it is the labor externality whose effect drives the result. This is not surprising, given the well known fact that the upward sloping labor demand curve is necessary for indeterminacy in this particular model.

Next we examine what happens when we allow for two sectors of production. Our results can be summarized as follows. We find that even with no externalities in consumption, indeterminacy results when the externalities from investment are sufficiently high. This result is similar to that of Harrison (1999) and Weder (1999). Furthermore, even with no externalities from labor, indeterminacy results when the externalities from capital are sufficiently high. We are able to prove these results analytically and to provide some intuition for them.

Finally, we merge the models and investigate the complementary effects of sector-specific and aggregate externalities in the two-sector model. We

[2]See Harrison (1999) for a model in which utility is of the more general constant relative risk aversion form.

find that adding aggregate externalities into the model does not reduce the overall increasing returns that are needed for indeterminacy. On the contrary, a simple trade-off between aggregate and sector-specific externalities is seen.

The rest of this paper proceeds as follows. Section 2 presents the model. In Section 3 we analyze the stability properties of the one sector model. Then we do the same with the two sector model in Section 4. In Section 5 we consider the general case with both kinds of externalities in the two sector model. Section 6 concludes.

2 The model

The model is similar to that of Benhabib and Farmer (1996), though it is less restrictive. The household derives utility from consumption and leisure; and firms produce with a constant returns to scale technology and an externality taken as given. All markets are perfectly competitive. Our model departs from that of Benhabib and Farmer in one significant way: we relax restrictions on the external effects in a two fold manner. First, we have both aggregate and sector-specific externalities. In this way, the economy can have only one sector of production, or two separate sectors that produce consumption and investment goods. That is, we actually have two different models. Second, we allow for factor-specific externalities. That is, externalities arising from labor may differ from those from capital. Most of our results come from the latter alteration to the model.

2.1 The household problem

The representative consumer maximizes the utility criterion

$$U = \int_0^\infty U\left(C_t, L_t\right)\ e^{-\rho t} dt \qquad \rho > 0$$

where C_t, L_t and ρ stand for consumption, labor, and the rate of pure time preference respectively. Let instantaneous utility be given by[3]

$$U\left(C_t, L_t\right) = \ln C_t - \psi L_t \qquad \psi > 0.$$

We assume the standard capital accumulation technology of the form

$$\dot{K}_t = I_t - \delta K_t \qquad \delta \in (0,1)$$

where I_t is the household's investment expenditures and δ denotes the rate of capital depreciation. Letting w_t denote the wage rate and r_t the rental rate on capital, the period income identity is given by

$$w_t L_t + r_t K_t = C_t + p_t I_t.$$

where p_t is the relative price of investment and the price of consumption is normalized to one. Denote by Λ_t the shadow value of wealth. Then, the agent's first order conditions are

$$C_t^{-1} = \Lambda_t p_t^{-1} \tag{1}$$

$$\psi = w_t \Lambda_t p_t^{-1} \tag{2}$$

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = \rho + \delta - r_t p_t^{-1} \tag{3}$$

where (2) describes the intratemporal leisure-consumption trade-off and (3) is the intertemporal Euler equation for consumption. In addition, the household must obey the usual transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \Lambda_t K_t = 0.$$

[3]We restrict the study to the case of logarithmic utility since it is consistent with balanced growth. Indivisible labor is standard in the Real Business Cycles literature.

2.2 The firms' problems

In the consumption sector, the firm maximizes profit subject to:

$$y_t^C = A_t {k_{c,t}}^{\alpha} {l_{c,t}}^{1-\alpha} \qquad \alpha \in (0,1)$$

where

$$A_t = {K_{c,t}}^{\alpha\theta_K^C} {L_{c,t}}^{(1-\alpha)\theta_L^C} K_t^{\alpha\gamma_K^C} L_t^{(1-\alpha)\gamma_L^C}.$$

Here $k_{c,t}$ and $l_{c,t}$ denote the capital and labor the firm devotes to production of the consumption good at time t and α is capital's share in production of the consumption good. $K_{c,t}$ and $L_{c,t}$ denote the economy-wide average capital and labor devoted to production of the consumption good, which are taken as given by the firm. Also taken as given are K_t and L_t, the economy-wide averages of total capital and labor. The parameters θ_K^C and θ_L^C denote the sizes of the sector- and factor-specific externalities while γ_K^C and γ_L^C denote the sizes of the aggregate factor-specific externalities. That is, the sector-specific externalities are benefits from the sector-specific inputs only while aggregate externalities come from all inputs. Recall that it is the distinction of externalities over different inputs that establishes a novel aspect of this paper.

In the investment sector, the firm maximizes profit subject to:

$$y_t^I = B_t {k_{I,t}}^{\alpha} {l_{I,t}}^{1-\alpha}$$

where

$$B_t = {K_{I,t}}^{\alpha\theta_K^I} {L_{I,t}}^{(1-\alpha)\theta_L^I} K_t^{\alpha\gamma_K^I} L_t^{(1-\alpha)\gamma_L^I}.$$

Here $k_{I,t}$ and $l_{I,t}$ denote the capital and labor the firm devotes to production of the investment good. $K_{I,t}$ and $L_{I,t}$ denote the economy-wide average capital and labor used in such production, which are taken as given. The parameters θ_K^I and θ_L^I denote the sizes of the sector- and factor-specific externalities while γ_K^I and γ_L^I denote the sizes of the aggregate factor-specific externalities.

Given their goals of maximizing profits, firms hire capital and labor to satisfy:

$$r_t = \frac{\alpha y_t^C}{k_{c,t}} = p_t \frac{\alpha y_t^I}{k_{I,t}} \tag{4}$$

and

$$w_t = \frac{(1-\alpha) y_t^C}{l_{c,t}} = p_t \frac{(1-\alpha) y_t^I}{l_{I,t}}. \tag{5}$$

Throughout the paper, we will assume that the following holds.

Assumption *The level of increasing returns from all sources is modest. That is,* $\alpha(1 + \theta_j^i + \gamma_j^i) < 1$ for $i = C,\ I$ *and* $j = K,\ L$.[4]

2.3 Equilibrium and Dynamics

In competitive symmetric equilibrium we have $k_{c,t} = K_{c,t}$ and $l_{c,t} = L_{c,t}$. Further, we can write

$$Y_t^C = \mu_t^{1+\alpha\theta_K^C + (1-\alpha)\theta_L^C} K_t^{\alpha(1+\theta_K^C+\gamma_K^C)} L_t^{(1-\alpha)(1+\theta_L^C+\gamma_L^C)}$$

and

$$Y_t^I = (1-\mu_t)^{1+\alpha\theta_K^I + (1-\alpha)\theta_L^I} K_t^{\alpha(1+\theta_K^I+\gamma_K^I)} L_t^{(1-\alpha)(1+\theta_L^I+\gamma_L^I)}$$

where $Y_t^C$and Y_t^I denote economy-wide average values of output of the two sectors and

$$\mu_t \equiv \frac{K_t^C}{K_t} = \frac{L_t^C}{L_t}.$$

[4]Our definition of modesty includes values of the externality that are empirically plausible given evidence in Basu and Fernald (1997) and Harrison (1998). Note that increasing returns are not high enough to induce endogenous growth.

Also, we can solve for the relative price of investment:

$$p_t = \frac{\mu_t^{\alpha\theta_K^C+(1-\alpha)\theta_L^C}}{(1-\mu_t)^{\alpha\theta_K^I+(1-\alpha)\theta_L^I}} K_t^{\alpha(\theta_K^C-\theta_K^I+\gamma_K^C-\gamma_K^I)} L_t^{(1-\alpha)(\theta_L^C-\theta_L^I+\gamma_L^C-\gamma_L^I)}. \tag{6}$$

We define a perfect foresight equilibrium as follows.[5]

Definition *A perfect foresight equilibrium is a set of sequences* $\{K_{t+1},\ L_t,\ \mu_t,\ \Lambda_t\}_{t=0}^{\infty}$ *and an initial capital stock* $K(0) > 0$ *satisfying (1) to (3), (4), and (5). In addition, the resource constraints and the transversality condition hold.*

The model does not have a closed form solution. Thus, we derive local dynamics by Taylor approximating around the steady state. The behavior of the model boils down to the two-dimensional system

$$\begin{bmatrix} \log \dot{\Lambda}_t \\ \log \dot{K}_t \end{bmatrix} = \mathbf{J} \begin{bmatrix} \log \Lambda_t - \log \Lambda \\ \log K_t - \log K \end{bmatrix} \tag{7}$$

where $\mathbf{J}$ is the 2×2 Jacobian. From (7), the equilibrium of the model is indeterminate if there exists an infinite number of (locally) perfect foresight equilibrium sequences of capital and the shadow price, given an initial capital stock. Recall that Λ_t is a non-predetermined variable and that K_t is predetermined. Thus, indeterminacy requires that both eigenvalues of $\mathbf{J}$ have negative real parts. Since the trace of the matrix is the sum of its eigenvalues and the determinant is the product of the eigenvalues, indeterminacy can be restated as

$$\mathrm{Tr}\mathbf{J} < 0 < \mathrm{Det}\mathbf{J}.$$

Similarly, the steady state is saddle path stable if $\mathrm{Det}\mathbf{J} < \mathbf{0}$ and it is a source if

$$\mathrm{Tr}\mathbf{J} > \mathbf{0} \quad \text{and} \quad \mathrm{Det}\mathbf{J} > \mathbf{0}.$$

[5]See the Appendix which discusses our solution mechanism in full detail.

In the next three sections we examine three different versions of the model. In the first version, there are no sector-specific externalities, only aggregate externalities. In this case, the model is similar to the one sector model of Benhabib and Farmer (1994) – in fact, theirs is a special case of the present model. In the second version, there are no aggregate externalities and the model has two sectors of production as in Benhabib and Farmer (1996), Harrison (1999) and Weder (1999). In the last version, we examine the complementary effects of sector-specific and aggregate externalities in the two sector model. Under various restrictions, we find analytic expressions for the trace and determinant of the **J** matrix for each model. This allows us to find conditions under which indeterminacy can result. Our findings tell us that in the one sector model externalities are required in both capital and labor while in the two sector model a necessary and sufficient condition for indeterminacy is that a certain size capital externality operate.

3 The one sector model

In this section we set $\theta_j^i = 0 \;\; for\; i = C,\, I \;\; and\; j = K,\, L$ but allow for positive values of the aggregate externalities. In this case, (6) becomes

$$p_t = 1$$

so that output, Y_t, is produced in only one sector and is given by

$$Y_t = Y_t^C + p_t Y_t^I = K_t^{\alpha(1+\gamma_K)} L_t^{(1-\alpha)(1+\gamma_L)}$$

where we have omitted superscripts on the externality parameters. The model reduces to the one sector model of Benhabib and Farmer (1994), except that they restrict $\gamma_K = \gamma_L$.

3.1 Stability properties

In this subsection, we examine the stability properties of this model as we change the externality parameters. We start with the case where $\gamma_K = \gamma_L = \gamma$. In this case,

$$\text{Det}\mathbf{J} = \frac{(\rho+\delta)(\rho+(1-\alpha)\delta)\left(\alpha(1+\gamma)-1\right)}{\alpha\left[\alpha-(1-\alpha)\gamma\right]}. \tag{8}$$

and

$$\text{Tr}\mathbf{J} = \frac{\alpha\rho+(\alpha\rho+\delta)\gamma}{\alpha-(1-\alpha)\gamma}$$

Given that we have assumed modest increasing returns ($\alpha(1+\gamma)<1$), the numerator of (8) is negative. Therefore, we have the following condition for indeterminacy.

Proposition 1 *The equilibrium of the one sector model is indeterminate if*

$$\gamma > \gamma^{\min} = \frac{\alpha}{1-\alpha}.$$

This condition is essentially the same as in Benhabib and Farmer (1994). Following the calibration suggested by Benhabib and Farmer (1996), i.e. $\alpha = 0.3$, $\delta = 0.1$, and $\rho = 0.05$, which will serve as our model calibration throughout the rest of the paper, $\gamma^{\min} = 0.4285$, which is clearly outside the empirically plausible range given evidence in Basu and Fernald (1997).

Next, we consider the case with $\gamma_K \neq \gamma_L$. The determinant and trace are given by

$$\text{Det}\mathbf{J} = \frac{(\rho+\delta)(\rho+(1-\alpha)\delta)\left(\alpha(1+\gamma_K)-1\right)}{\alpha\left[\alpha-(1-\alpha)\gamma_L\right]}.$$

and

$$\text{Tr}\mathbf{J} = \frac{\alpha\rho+(\rho+\delta)\gamma_K-(1-\alpha)\rho\gamma_L}{\alpha-(1-\alpha)\gamma_L}$$

The determinant is positive for

$$\gamma_L > \frac{\alpha}{1-\alpha}, \tag{9}$$

which is equal to $\gamma^{\min}$, and we denote it $\gamma_L^{\min}$ from here on. Interestingly, when we assume factor-specific aggregate externalities, the level of externality necessary for indeterminacy stays the same as when $\gamma_K = \gamma_L = \gamma$. For sufficiency, we also need the trace to be negative. Assuming (9) holds, a negative trace requires

$$\gamma_K > \frac{\rho}{\rho+\delta}((1-\alpha)\gamma_L - \alpha), \tag{10}$$

which is positive but infinitesimally close to zero as γ_L approaches $\gamma_L^{\min}$. We are able to formulate the following Proposition.

Proposition 2 *Indeterminacy arises in the one sector model if and only if*

$$\frac{(\rho+\delta)\gamma_K + \alpha\rho}{\rho(1-\alpha)} > \gamma_L > \frac{\alpha}{1-\alpha}.$$

Figure 1 illustrates the stability properties of this model. With γ_L on the horizontal axis and γ_K on the vertical axis, we see how the equilibrium changes when $\gamma_L > \gamma_L^{\min}$. When $\gamma_K > \rho((1-\alpha)\gamma_L - \alpha)/\left(\rho+\delta\right)$, the move is from determinacy to indeterminacy at $\gamma_L^{\min}$. On the other hand, as γ_L increases, and if instead $\gamma_K < \rho((1-\alpha)\gamma_L - \alpha)/\left(\rho+\delta\right)$, the system becomes a source.

The one sector model is determinate for sufficiently low levels of externalities that arise from aggregate labor. Therefore, the indeterminacy mechanism appears to founded in the labor market, reflecting the now well known story that indeterminacy results because of an upward sloping labor demand curve. However, Proposition 2 restricts the parameter space for indeterminacy even further. Basically, given (nonnegative) capital externalities, γ_L must not be too high, otherwise the system becomes a source. In a sense, it is a positive valued γ_K that acts to stabilize the model's dynamics in the strict indeterminacy sense. Thus, for the one sector model to generate indeterminacy, increasing returns must be primarily associated with labor.

4 The two sector model

In this section we set $\gamma_j^i = 0$ for $i = C,\ I\ \textit{and}\ j = K,\ L$ but allow for positive sector-specific externalities. In this case, the model reduces to a more general version of that in Benhabib and Farmer (1996). There are now two distinct sectors of production: consumption and investment.

As above, we find expressions for the determinant and trace of the $\mathbf{J}$ matrix and conditions for indeterminacy. First we restrict $\theta_K^I = \theta_L^I = \theta^I$ and find that indeterminacy can result with θ^I above a certain value. That is, indeterminacy can result in the two sector model even with no externalities in consumption. Next we find that indeterminacy can result with no externality from labor, as long as the externality from capital is above a certain minimum. However, the opposite is not true. Finally we are able to pin down the indeterminacy region while allowing both externalities to be positive.

4.1 Stability properties

First, we restrict $\theta_K^I = \theta_L^I = \theta^I$. We find:

$$\mathrm{Det}\mathbf{J} = \frac{\delta\left(\rho+\delta\right)\left(\rho+(1-\alpha)\delta\right)(\alpha(1+\theta^I)-1)}{\alpha^2\delta+\theta^I\left(\alpha^2\delta-\rho-\delta\right)}$$

and

$$\mathrm{Tr}\mathbf{J} = \frac{\alpha^2\rho\delta+(\alpha\delta^2+\rho\left(\alpha\delta(1+\alpha)-\rho-\delta\right))\theta^I}{\alpha^2\delta+\theta^I\left(\alpha^2\delta-\rho-\delta\right)}.$$

Note that neither θ_K^C nor θ_L^C appear in these expressions. That is, consumption externalities do not affect the stability properties when utility is logarithmic. This result is reminiscent of Harrison (1999) and Weder (1999). It can be understood as follows. As returns to scale in consumption increase, consumers will want to smooth consumption due to risk aversion. On the other hand, in order to take advantage of the increasing returns to scale, they might prefer volatile consumption. These two effects work against each other, but the higher the risk aversion, the more the desire to smooth will dominate. With logarithmic utility, the two effects exactly cancel.

Imposing $\alpha(1+\theta^I) < 1$, the numerator of Det$\mathbf{J}$ is negative. Now, the conditions for indeterminacy are summarized as follows.

Proposition 3 *In the two sector model, when $\theta_K^I = \theta_L^I = \theta^I$, indeterminacy results if and only if*

$$0 < \frac{\alpha^2\delta}{\rho + (1-\alpha^2)\delta} < \theta^I < \frac{-\alpha^2\delta\rho}{\alpha\delta^2 - \rho\left(\rho + \delta - (1+\alpha)\alpha\delta\right)}$$

when $\alpha\delta^2 < \rho\left(\rho + \delta - (1+\alpha)\alpha\delta\right)$ or

$$0 < \frac{\alpha^2\delta}{\rho + (1-\alpha^2)\delta} < \theta^I$$

otherwise.

For the above values of θ^I, the determinant is positive and the trace is negative. Our model calibration falls into the first category, and denoting the minimum and maximum investment-sector-specific externalities $\theta^I_{\min}$ and $\theta^I_{\max}$ we have $\theta^I_{\min} = 0.0638$ and $\theta^I_{\max} = 0.1764$. Within this range are empirically plausible values. Above the second value, the steady state is a source[6]. Note, however, that for some calibrations the upper bound does not matter

[6] At that point the model has a simple pair of imaginary eigenvalues and a Hopf bifurcation takes place. The corresponding analysis is beyond the scope of this paper.

since the trace is always negative for positive θ^I. This latter case corresponds to the second part of Proposition 3.[7]

Next, allowing factor-specific externalities to differ, e.g. $\theta^I_K \neq \theta^I_L$, the eigenvalues are implicitly described by

$$\mathrm{Det}\mathbf{J} = \frac{\delta\,(\rho+\delta)\,(\rho+(1-\alpha)\delta)(\alpha(1+\theta^I_K)-1)}{\alpha^2\delta-\alpha\,(\rho+(1-\alpha)\delta)\,\theta^I_K-(1-\alpha)\,(\rho+\delta)\,\theta^I_L}$$

and

$$\mathrm{Tr}\mathbf{J} = \frac{\alpha^2\rho\delta+\alpha(\delta^2-\rho^2+\alpha\delta\rho)\theta^I_K-\rho(1-\alpha)\,(\rho+\delta)\,\theta^I_L}{\alpha^2\delta-\alpha\,(\rho+(1-\alpha)\delta)\,\theta^I_K-(1-\alpha)\,(\rho+\delta)\,\theta^I_L}.$$

We first set $\theta^I_K = 0$. In this case,

$$\mathrm{Tr}\mathbf{J} = \rho > 0$$

so that the equilibrium cannot be indeterminate. If, however we set $\theta^I_L = 0$, we find that indeterminacy arises for

$$\theta^I_K > \frac{\alpha\delta}{\rho+(1-\alpha)\delta}.$$

Therefore, we have

Proposition 4 *In the two sector model with $\theta^I_K = 0$, indeterminacy cannot result.*

Proposition 5 *In the two sector model with $\theta^I_L = 0$, indeterminacy results if and only if*

$$\theta^I_K > \theta^I_{K\min} = \frac{\alpha\delta}{\rho+(1-\alpha)\delta}.$$

Under our model calibration, $\theta^I_{K\min} = 0.25$, which is about four times larger than in the case where sector-specific externalities from labor were

[7]This case can appear for larger capital shares and lower discount rates than in our model calibration.

included as well. That is, despite the fact that externalities in labor cannot generate indeterminacy alone, without them, the externality from capital must be stronger. This makes clear that there is a complementary relationship between the two externalities in producing indeterminacy[8].

It is appropriate at this point to compare the values of $\gamma_L^{\min}$, $\theta_{\min}^I$, and $\theta_{K\min}^I$. We have the following corollary:

Corollary *In the models, minimum scale economies that imply indeterminacy* $\left(\gamma_L^{\min}, \theta_{\min}^I, \theta_{K\min}^I\right)$ *are such that*

$$\theta_{\min}^I < \theta_{K\min}^I < \gamma_L^{\min}$$

for every parameter constellation (α, δ, ρ).

Proof. Suppose that the left hand side inequality does not hold. Then,

$$\alpha\left(\rho + (1-\alpha)\delta\right) \geq \rho + (1-\alpha^2)\delta$$

or

$$(\rho + \delta)(\alpha - 1) \geq 0$$

which clearly is contradicted by the assumptions made on the model parameters. In addition, if we suppose that the second inequality does not hold, we have

$$\delta(1-\alpha) > \rho + (1-\alpha)\delta$$

or

$$0 > \rho$$

which is also contradicted. ■

[8] It is of interest to note that at $\theta_K^I = (1-\alpha)/\alpha$, the economy transforms into the Ak-style growth model. In other words, in the two sector model long-run growth can originate from sector-specific externalities that are confined to capital alone.

The above demonstrates that the minimum externalities needed for indeterminacy are always larger in the one sector than in the two sector model, and that leaving out the labor externality raises the required capital externality in the two sector model.

Lastly, we experiment by allowing both externalities to be simultaneously positive. By solving for the trace and determinant, we can formulate our next Proposition.[9]

Proposition 6 *In the two sector model, indeterminacy results if only if*

$$-\alpha\rho\left(\rho+(1-\alpha)\delta\right)\theta_K^I<\rho\left(1-\alpha\right)\left(\rho+\delta\right)\theta_L^I-\alpha^2\delta\rho<\alpha\left(\delta^2-\rho^2+\alpha\delta\right)\theta_K^I.$$

Figure 2 summarizes the different regions that arise in the $\theta_K^I-\theta_L^I-$plane. Let us examine the two portions of the curve. The downward sloping region corresponds to $\theta_K^I-\theta_L^I$-combinations at which the determinant transfers from negative into positive territory, while the trace is negative. In other words, there is a move from determinacy to indeterminacy. As the labor externality increases, the size of the capital externality needed for indeterminacy decreases. It seems that agents look for some overall level of increasing returns, so that a trade-off exists between the sectoral values. The upward sloping portion corresponds to $\theta_K^I-\theta_L^I$-combinations at which the trace transfers from negative into positive territory as we increase θ_L^I. Along here, the equilibrium changes from indeterminate to a source. Three results obtain: (1) above a certain level of θ_L^I, determinate equilibria cannot result; (2) given a value of θ_L^I, lower values of θ_K^I result in a source while higher values of θ_K^I result in indeterminacy; and (3) as the labor externality increases, the size of the capital externality needed to move from a source to indeterminacy increases. It is interesting to note that these results imply that given a value

[9]The results here require $\rho<\delta$, which assumption holds under reasonable parameterizations. Without this assumption, the indeterminacy region may become even larger.

of θ_K^I, as θ_L^I increases, instability can result. However, given a value of θ_L^I, as θ_K^I increases, the equilibrium of the model remains indeterminate.

The economic intuition for the stability behavior here is closely linked to the indeterminacy mechanism of the two sector model. Essentially, in the two sector model multiple equilibria are the consequence of self-fulfilling sequences of the relative price of investment goods. Given a modest amount of increasing returns θ_K^I, increases in investment activity generate a decline in that price. Thus, by shifting expenditures over time but operating along an optimal path, the household initially induces a fall and then a rise in the relative price. It is these self-fulfilling capital gains that constitute indeterminacy in the two sector model. Also, labor-related externalities contribute to indeterminacy as changes in that input shift the returns to capital, thus helping the realization of mentioned capital gains. However, externalities that are linked to labor primarily affect the labor market. They do not directly contribute to the above countercyclical pricing sequence, in fact when considered alone they appear to be counteractive – interestingly, here the one and the two sector model observe some equivalence, since it is always the externalities associated with labor that induce the steady state transformation from sink to source.

5 The two sector model with aggregate externalities

In this section we assume the presence of both sector-specific and aggregate externalities. Now we have

$$\mathrm{Det}\mathbf{J} = \frac{\delta\left(\rho+\delta\right)\left(\rho+(1-\alpha)\delta\right)(\alpha(1+\gamma_K^I+\theta_K^I)-1)}{\alpha^2\delta-\alpha\left(\rho+(1-\alpha)\delta\right)\theta_K^I-(1-\alpha)\left(\rho+\delta\right)\theta_L^I-\alpha\delta(1-\alpha)\gamma_L^I}$$

and

$$\mathrm{Tr}\mathbf{J} = \frac{\alpha^2\rho\delta + \alpha(\delta^2 - \rho^2 + \alpha\delta\rho)\theta_K^I - \rho(1-\alpha)\left(\rho+\delta\right)\theta_L^I + \alpha\delta(\rho+\delta)\gamma_K^I - \alpha\delta\rho(1-\alpha)\gamma_L^I}{\alpha^2\delta - \alpha\left(\rho + (1-\alpha)\delta\right)\theta_K^I - (1-\alpha)\left(\rho+\delta\right)\theta_L^I - \alpha\delta(1-\alpha)\gamma_L^I}.$$

We note that γ_K^I has no effect on the determinant. Thus, it does not influence the necessary condition for indeterminacy. Given the results in Propositions 4 to 6, as seen in Figure 2, we can disregard the effects of θ_L^I. Therefore, we restrict our analysis to simply θ_K^I and γ_L^I. From the condition for a positive determinant, we can derive the following Proposition.[10]

Proposition 7 *In the two sector model when $\theta_L^I = 0$, indeterminacy arises when*

$$\theta_K^I > \frac{\alpha\delta - (1-\alpha)\delta\gamma_L^I}{\rho + (1-\alpha)\delta}.$$

First of all, observe that neither consumption sector externality appears in these expressions. Further, note that when $\theta_K^I = 0$, this condition becomes $\gamma_L^I > \frac{\alpha}{1-\alpha}$, which is the same as $\gamma_L^{\min}$ from the analysis of the one sector model. When $\gamma_L^I = 0$, we need $\theta_K^I > \frac{\alpha\delta}{\rho+(1-\alpha)\delta}$,which is the same as $\theta_{K\min}^I$ from the analysis of the two sector model. Allowing for both types of externalities does not reduce the minimum overall amount needed for indeterminacy. One way to see this is to observe that

$$0 > \frac{\partial\theta_{K\min}^I(\gamma_L^I)}{\partial\gamma_L^I} = \frac{-(1-\alpha)\delta}{\rho + (1-\alpha)\delta} > -1.$$

In Figure 3, we depict the line between determinacy and indeterminacy. With γ_L^I on the horizontal axis and θ_K^I on the vertical axis, the intercepts are the same as in Figures 1 and 2 respectively. The downward sloped line shows the trade-off between sector-specific and aggregate effects. Essentially, as we move down the negatively sloped line, the model is changing from one with

[10]It is generally the case that the trace is negative at the bifurcation point.

two sectors to one with only one sector. The "45 degree line" is also drawn to emphasize the fact that adding in aggregate externalities does not reduce the level of overall externalities needed for indeterminacy.

6 Conclusion

This paper offers further inquiry into sources for indeterminacy in general equilibrium. We take a more agnostic path than previous work as we allow for a more general setup and interpretation of externalities. In particular, we identify the origin of scale economies (either from labor or capital) that induces multiple equilibria. We find that in the one-sector model it is the externalities associated with labor that generate the result. Matters are different in the two-sector model, however. Here indeterminacy can be ultimately traced back to the external effects from capital in the investment sector. It is these externalities that set in motion the countercyclical pricing sequence that constitutes indeterminacy in the two sector model. We also show that versions of the model that consider both kinds of externalities do not reduce the level of overall externalities needed for multiple equilibria. In fact, it is the two sector model with sector-specific externalities alone that obtains indeterminacy at minimal departures from constant returns.

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7 Appendix (not to be published)

7.1 Dynamics of one sector model

The model is completely described by the following three equations.[11] Labor market clearing implies

$$\psi = \Lambda_t(1-\alpha)K_t^{\alpha(1+\gamma_K)}L_t^{(1-\alpha)(1+\gamma_L)-1}. \tag{11}$$

The shadow value of wealth Λ_t must obey

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = \delta + \rho - \alpha K_t^{\alpha(1+\gamma_K)-1}L_t^{(1-\alpha)(1+\gamma_L)} \tag{12}$$

and capital accumulates according to

$$\frac{\dot{K}_t}{K_t} = K_t^{\alpha(1+\gamma_K)-1}L_t^{(1-\alpha)(1+\gamma_L)} - \delta - \Lambda_t^{-1}K_t^{-1}. \tag{13}$$

It is easily shown that the steady state is unique. It implies

$$\frac{Y}{K} = \frac{\rho+\delta}{\alpha}.$$

Here, the time subscript has been dropped to indicate stationary values. The associated Jacobian matrix $\mathbf{J}$ is given by

$$\mathbf{J} = \begin{bmatrix} -\frac{(\delta+\rho)(1-\alpha)(1+\gamma_L)}{\alpha-(1-\alpha)\theta_L} & -\frac{(\delta+\rho)(\alpha\gamma_K+(1-\alpha)\gamma_L)}{\alpha-(1-\alpha)\theta_L} \\ \frac{\delta+\rho-\alpha^2\delta+\alpha\delta(1-\alpha)\gamma_L}{\alpha(\alpha-(1-\alpha)\gamma_L)} & \frac{(\delta+\rho)(1+\gamma_K)-\alpha\delta+(1-\alpha)\delta\gamma_L}{\alpha-(1-\alpha)\gamma_L} \end{bmatrix}.$$

7.2 Dynamics of two sector model

The set of the following four equations constitutes the two sector dynamics. Labor market clearing is described by

$$\psi = \Lambda_t(1-\alpha)(1-\mu_t)^{\alpha\theta_K^I+(1-\alpha)\theta_L^I}K_t^{\alpha(1+\theta_K^I)}L_t^{(1-\alpha)(1+\theta_L^I)-1}. \tag{14}$$

[11] Recall that we assume that $\sigma = 1$.

The first order condition for consumption and the production function of the consumption good together imply

$$\mu_t^{\left(\alpha\theta_K^C+(1-\alpha)\theta_L^C\right)(1-\sigma)-\sigma}(1-\mu_t)^{-\alpha\theta_K^I-(1-\alpha)\theta_L^I} \tag{15}$$
$$= \Lambda_t K_t^{\alpha\sigma(1+\theta_K^I)-\alpha(\theta_K^C-\theta_K^I)} L_t^{(1-\alpha)\sigma(1+\theta_K^I)-(1-\alpha)(\theta_K^C-\theta_K^I)} \tag{16}$$

The shadow value of wealth Λ_t must obey

$$\frac{\dot{\Lambda}_t}{\Lambda_t} = \delta + \rho - \alpha(1-\mu_t)^{\alpha\theta_K^I+(1-\alpha)\theta_L^I} K_t^{\alpha(1+\theta_K^I)-1} L_t^{(1-\alpha)(1+\theta_L^I)} \tag{17}$$

and capital accumulates according to

$$\frac{\dot{K}_t}{K_t} = (1-\mu_t)^{1+\alpha\theta_K^I+(1-\alpha)\theta_L^I} K_t^{\alpha(1+\theta_K^I)-1} L_t^{(1-\alpha)(1+\theta_L^I)} - \delta. \tag{18}$$

The two intertemporal equations entail in steady state that

$$1-\mu = \frac{\alpha\delta}{\delta+\rho}.$$

The steady state is unique. The loglinearized model consist of the two equations

$$\mathbf{\Pi}_1 \begin{bmatrix} \log\Lambda_t - \log\Lambda \\ \log K_t - \log K \end{bmatrix} = \mathbf{\Pi}_2 \begin{bmatrix} \log L_t - \log L \\ \log\mu_t - \log\mu \end{bmatrix}$$

and

$$\begin{bmatrix} \dot{\log\Lambda_t} \\ \dot{\log K_t} \end{bmatrix} = \mathbf{J}_1 \begin{bmatrix} \log\Lambda_t - \log\Lambda \\ \log K_t - \log K \end{bmatrix} + \mathbf{J}_2 \begin{bmatrix} \log L_t - \log L \\ \log\mu_t - \log\mu \end{bmatrix}.$$

Linearizing the static equations lead to

$$\mathbf{\Pi}_1 = \begin{bmatrix} 1 & \alpha\left(1+\theta_K^I\right) \\ 1 & \alpha\left(1+\theta_K^I\right) \end{bmatrix}$$

and

$$\mathbf{\Pi}_2 = \begin{bmatrix} -1+(1-\alpha)\left(1+\theta_L^I\right) & -\alpha\theta_K^I-(1-\alpha)\theta_L^I \\ (\alpha-1)\left[\theta_L^C-\theta_L^I-\sigma\left(1+\theta_L^C\right)\right] & -\alpha\theta_K^I-(1-\alpha)\theta_L^I+\frac{(1-\mu)\sigma}{\mu}\,\square\,\frac{\left(\alpha\theta_K^I-(1-\alpha)\theta_L^I\right)(1-\sigma)}{\sigma}-1] \end{bmatrix}.$$

The two intertemporal equations imply

$$\mathbf{J}_1 = \left[\begin{array}{cc} 0 & -(\delta+\rho)(\alpha\left(1+\theta_K^I\right)-1) \\ 0 & \delta(\alpha\left(1+\theta_K^I\right)-1) \end{array} \right]$$

and

$$\mathbf{J}_2 = \left[\begin{array}{cc} -(\delta+\rho)(1-\alpha)\left(1+\theta_L^I\right) & -(\delta+\rho)(\alpha\theta_K^I(1-\alpha)\theta_L^I) \\ \delta(1-\alpha)\left(1+\theta_L^I\right) & \delta\left(1+\alpha\theta_K^I(1-\alpha)\theta_L^I\right) \end{array} \right].$$

The model reduces to

$$\left[\begin{array}{c} \dot{\log}\,\Lambda_t \\ \dot{\log}\,K_t \end{array} \right] = \mathbf{J} \left[\begin{array}{c} \log\Lambda_t - \log\Lambda \\ \log K_t - \log K \end{array} \right] \tag{19}$$

with

$$\mathbf{J} = \mathbf{J}_1 + \mathbf{J}_2 \mathbf{\Pi}_2^{-1} \mathbf{\Pi}_1$$

where $\mathbf{J}$ denotes the 2×2 Jacobian matrix of the linearized dynamical system.

Figure 1: Stability Properties of the One Sector Model

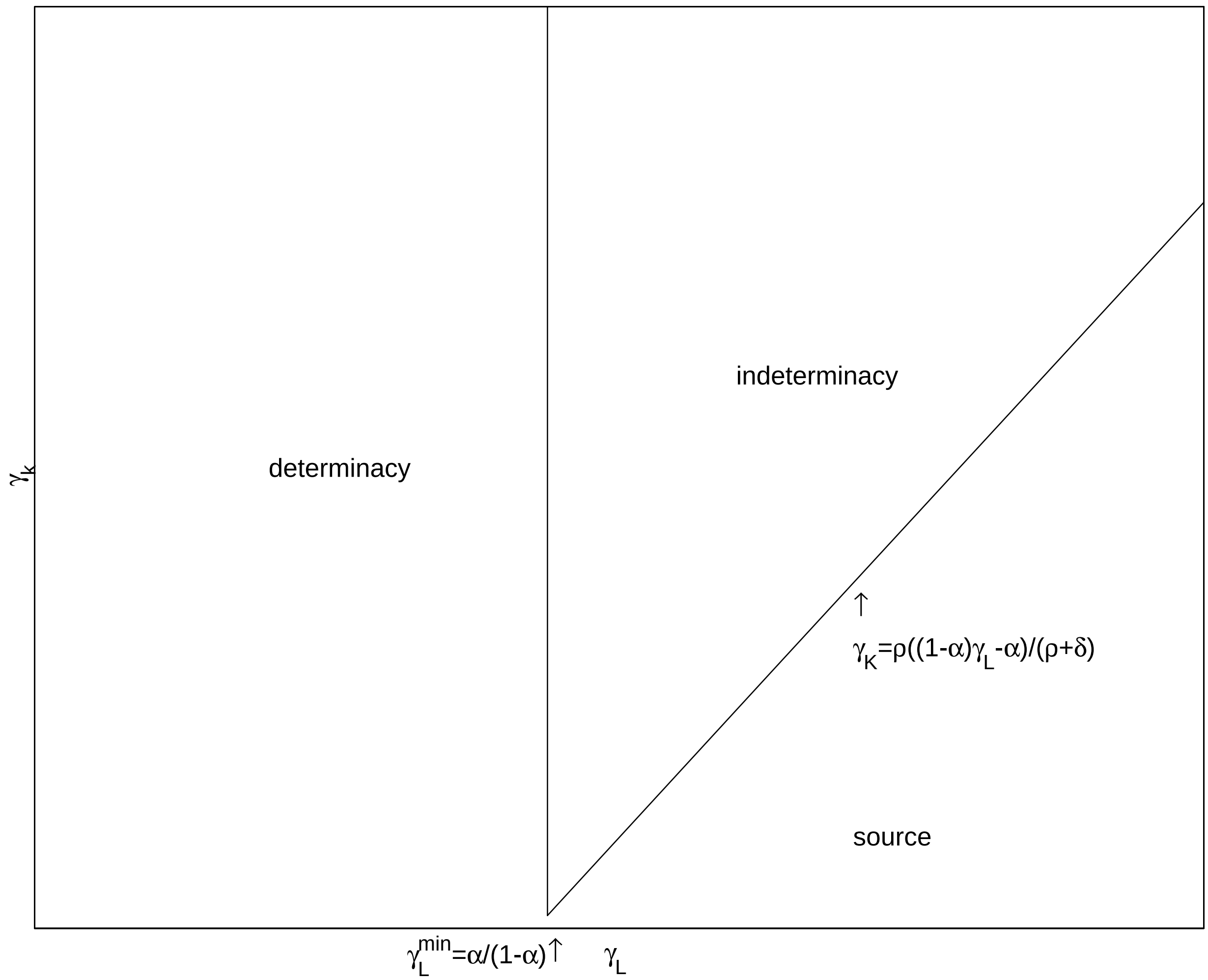

Figure 2: Stability Properties of the Two Sector Model

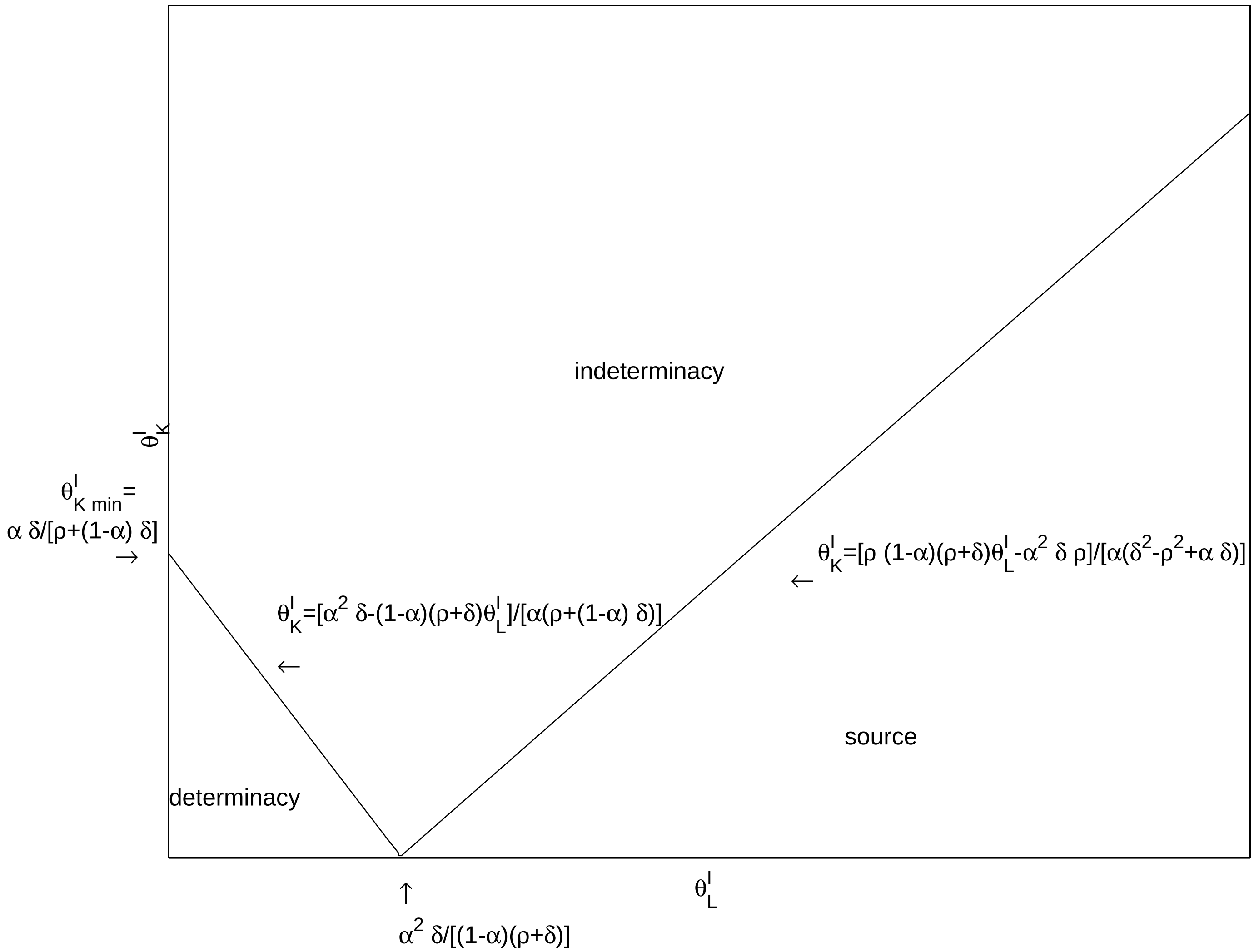

Figure 3: Stability Properties of the Two Sector Model with Aggregate Externalities